

an unbroken curve first, a well-defined tangent second: neither one is automatic

Continuity at a point

condition 1: $f(a)$ is defined, a real number actually comes out $f(a)$

condition 2: the limit approached with x below a matches the limit approached with x above a $LHL = RHL$

condition 3: that common limit equals the function's own value $LHL = RHL = f(a)$ at a

drop any one of the three and continuity fails at that point, even if the other two hold perfectly

Algebra of continuous functions

f and g both continuous at a : their sum, difference and product are too $f+g, f-g, fg$

their quotient is continuous at a as well, needs $g(a)$ not zero $\frac{f}{g}$

a continuous function of a continuous function stays continuous $g(f(x))$

every polynomial is continuous everywhere it's built purely from sums and products of x ; the quotient rule is where a denominator can quietly break things at its own zeros

Three ways continuity fails at $x=a$ *

check the limit and the function value at a

LHL and RHL don't match

no limit at all
the two-sided limit does not exist

limit exists, $f(a)$ undefined

a hole in the domain
there's nothing plotted at $x=a$ to compare against

limit and $f(a)$ disagree

a mismatched value
the curve jumps to a different height right at a

Check continuity of a piecewise function at $x=1$

Q. $f(x) = x^2$ for $x \leq 1$, and $f(x) = 2x-1$ for $x > 1$. Is f continuous at $x=1$?

- ① $f(1) = 1^2 = 1$, using the x^2 branch since $1 \leq 1$
- ② limit from below: as x approaches 1 with $x \leq 1$, x^2 approaches $1^2 = 1$
- ③ limit from above: as x approaches 1 with $x > 1$, $2x-1$ approaches $2(1)-1 = 1$
- ④ both one-sided limits equal 1, so the two-sided limit exists and equals 1
- ⑤ compare: the limit (1) matches $f(1)$ (1)

Answer f is continuous at $x=1$

Differentiability

- f is differentiable at a when the limit of $[f(a+h)-f(a)]/h$, as h approaches 0, exists
- that same limit must be identical whether h approaches 0 from positive or negative values
- differentiable at a point always forces continuous there: no exception to this direction
- converse fails: $|x|$ is continuous at $x=0$ but not differentiable; one-sided limits give -1 and $+1$

Continuous vs differentiable at a point *

Continuous



$f(x) = |x|$: an unbroken V, continuous everywhere including the corner at the middle

- just needs $LHL = RHL = f(a)$
- a corner, like $|x|$ at 0, is still continuous

Differentiable

- needs the same slope approaching from both sides
- always implies continuous; the converse fails at a corner

$|x|$ is the standing example: continuous at $x=0$, but its two one-sided slopes disagree there