

the chain rule run in every direction: through a composite, backward through an inverse, sideways through an implicit or parametric pair

Derivative of a Composite Function *

- a composite is a function of a function: $\sin(\log x)$, $e^{\tan x}$, $\log(x^5+4)$
- chain rule: $y=f(u)$, $u=g(x)$, both differentiable, then $dy/dx = (dy/du)(du/dx)$
- three deep, $y=f(v)$, $v=g(u)$, $u=h(x)$: $dy/dx = (dy/dv)(dv/du)(du/dx)$
- in practice: treat the inner block as one unit, differentiate the outer, multiply by its own derivative
- a product of composites, e.g. $(x^3+2x-3)^4(x+\cos x)^3$: product rule first, chain rule inside each factor
- one factor at a time, $u'v+uv'$, is where a chain-inside-a-product problem starts

Standard composite derivatives *

power of a function

$$\frac{d}{dx} [f(x)]^n = n[f(x)]^{n-1}f'(x)$$

sine of a function

$$\frac{d}{dx} \sin[f(x)] = \cos[f(x)] \cdot f'(x)$$

exponential of a function

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$$

log of a function

$$\frac{d}{dx} \log[f(x)] = \frac{f'(x)}{f(x)}$$

same chain rule, outer function swapped; at $n=1$ it's just $f'(x)$; $f'(x)$ multiplies in every time

Special cases in differentiation

$f(x)$ is a constant k → the derivative is zero, chain rule or not $\frac{d}{dx}(k)=0$

$|x|=1$ exactly, for $\sin^{-1}x$ or $\cos^{-1}x$ → the tangent is vertical, the derivative is undefined

$-1 < x < 1$, for $\sec^{-1}x$ or $\operatorname{cosec}^{-1}x$ → the function itself is undefined there, not just its derivative

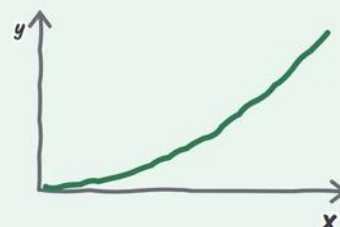
Chain rule through a square root

Q. Differentiate $y=\sqrt{x^2+5}$ w.r.t. x .

① treat x^2+5 as one block u , so $y=\sqrt{u}$

② $d/dx[\sqrt{u}] = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}$, and $du/dx = 2x$

③ $dy/dx = \frac{1}{2\sqrt{x^2+5}} \cdot 2x$



the right half of $y=\sqrt{x^2+5}$: a curve rising away from its minimum at $x=0$

Answer $dy/dx = x/\sqrt{x^2+5}$

Chain rule stacked two deep

Q. Differentiate $y=e^{\tan x}$ w.r.t. x .

① outer function $e^{(\cdot)}$, inner block $\tan x$

② $d/dx[e^{\tan x}] = e^{\tan x} \cdot \frac{d}{dx}(\tan x)$

③ $d/dx(\tan x) = \sec^2 x$

Answer $dy/dx = e^{\tan x} \sec^2 x$